

NONLOCAL MATHEMATICS

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Dedicated to Prof. Phillip E. Parker

ABSTRACT. This is a research report on what is best termed ‘nonlocal’ methods in mathematics. (This is not to be confused with global analysis.) The nonlocal formulation of physics in [PP] points to a fresh viewpoint in mathematics: the **nonlocal** viewpoint. It involves analyzing objects of geometry and analysis using nonlocal methods, as opposed to the classical local methods, e.g., Newton’s calculus. It also involves analyzing new nonlocal geometries and nonlocal analytical objects, i.e. nonlocal fields. In geometry, we introduce and study (nonlocal) forms, differentials, integrals, connections, curvatures, holonomy, **G**-structures, etc. In analysis, we analyze local fields using nonlocal methods (semilocal analysis); nonlocal fields nonlocally (of course); and the connection between nonlocal linear analysis and local *nonlinear* analysis. Analysis and geometry are next synthesized to yield nonlocal (hence noncommutative) homology, cohomology, de Rham theory, Hodge theory, Chern-Weil theory, **K**-theory (called **N**-theory) and index theory. Applications include theorems such as nonlocal- noncommutative Riemann-Roch, Gauss-Bonnet, Hirzebruch signature, etc. The nonlocal viewpoint is also investigated in algebraic geometry, analytic geometry, and, to some extent, in arithmetic geometry, resulting in powerful bridges between the classical and nonlocal-noncommutative aspects.

INTRODUCTION

It is propounded in [PP] that the basic fields of physics are nonlocal (Sec.1) and their local aspects are nonlocally related. Thus, laws of physics are formulated using a nonlocal calculus. This circumstance has led to an elaborate and fruitful development of what we term **nonlocal** geometry and analysis. These are not to be confused with global analysis and global aspects of differential geometry. Our fields are nonlocal in the sense that they are not defined point-wise, but rather at ordered pairs of points and their values are homomorphisms from the tangent space of the first point to that of the second. This initiates three lines of thought. First one leads to a nonlocal differential geometry (Sec.1); the second one leads to a nonlocal analysis of usual, local (point-wise), fields (Sec.2); and the third leads to an analysis of the nonlocal fields themselves (Sec.2).

Although all these considerations arose out of the problems of physics, the work [NM], the subject matter of the present report, naturally focuses on the mathematics itself. The threads mentioned above ([NM], Parts I & II) come together

Received by the editors February 25, 2000.

1991 *Mathematics Subject Classification.* Primary 00, Secondary 11, 14, 16, 19, 32, 35, 53, 55, 57R, 58, 81, 83.

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and interact fruitfully, to exhibit a very exciting interplay. The outcome ([NM], Part III, and Sec.3 below) is a set of nonlocal theories: noncommutative cohomology, nonlocal de Rham theory, nonlocal Hodge theory, nonlocal Chern-Weil theory, nonlocal (noncommutative) \mathbf{K} -theory, and nonlocal index theory. This last yields nonlocal analogs of various classical theorems such as Gauss-Bonnet-Chern, Hirzebruch-Riemann-Roch, Hirzebruch signature, Lefschetz fixed point, etc. Nonlocal methods carry over also to algebraic geometry and analytic geometry: deep connections are developed between commutative algebraic geometry and the geometry of non-commutative rings ([NM], Part IV, and Sec.4 below). Analytic geometry, too, benefits from nonlocal methods and accompanying noncommutative cohomology of sheaves ([NM], Part V, and Sec.5 below), and so does arithmetic geometry ([NM], Part VI, and Sec.6 below).

1. GEOMETRY

A **nonlocal vector (resp. affine) field** ω on a manifold X is an assignment to each ordered pair of points (x, y) a linear (resp. affine) homomorphism ω_{xy} from the tangent space T_x to the tangent space T_y . We call such a field a (nonlocal) **1-form**. Similarly, we can define a **(nonlocal) p -form** to be an assignment of a set of homomorphisms along the edges of each singular p -cube in X .

A **1-form** with invertible homomorphisms can also be thought of as a **nonlocal connection**: a connection, because it provides a means of comparing vectors at different points; **nonlocal** in the sense that the comparison is of vectors at *distinct* points, as opposed to the usual connection from differential geometry—which is essentially a means of comparing vectors at a point with those in its *infinitesimal* vicinity. The latter, too, enables comparison of vectors at distinct points, but the comparison depends on the path along which vectors are parallelly transported—which again underscores the local nature of these (classical) connections. Finally, the covariant exterior derivative defined by the classical connections is a local differential operator in a very precise sense. If two fields agree on a neighborhood, so do their covariant derivatives. The connection we are considering here is nonlocal also in that the basic “differential” operators associated with it are not local in the sense just mentioned.

Given a nonlocal connection ω , we define a **derivative $D\psi$ with respect to ω** of a p -form ψ , and then define the curvature of ω to be $D\omega$. A nonlocal connection ω has a local, classical, aspect. It is a usual, infinitesimal, connection and we denote it by the same letter ω . We call a nonlocal connection **flat** if its local aspect is flat. Then we have a theorem which says that a nonlocal connection ω is flat if and only if $D\omega = 0$. Also, a nonlocal analog of Bianchi identity holds: $D^2\omega = 0$.

We also have a notion of plain **derivative** of a p -form without any reference to a connection. This derivative is denoted by d . It satisfies $d^2 = 0$. A nonlocal notion of integration is introduced, and a nonlocal analog of the Stoke’s theorem is also obtained. The latter is perhaps best described as **the fundamental theorem of nonlocal calculus**.

All these notions can be made precise in the language of bundles. In particular, we can define **p -forms with values in an arbitrary vector bundle**. Also, all the notions of differential geometry have obvious analogs in this nonlocal setting. Thus, besides connections, we also have nonlocal metric structures, \mathbf{G} -structures, nonlocal holonomy, nonlocal structure equations, etc.

2. ANALYSIS

Analysis in the nonlocal setting takes two forms.

The first analyzes the usual local (point-wise) fields using nonlocal connections. This entails the introduction of a nonlocal calculus of local fields. (We call this **semilocal analysis**: local fields, nonlocal analysis.) Here, too, we have natural notions of (nonlocal) integral and derivative. These two are related by **the fundamental theorem of semilocal calculus**, which says that these two notions are inverses of each other. Like its Newtonian predecessor, it is *the* core of the calculus. Then, we analyze nonlocal differential operators, on spaces of sections of vector bundles. The theory is very similar in spirit to the usual analysis. Also, the linear semilocal analysis is closely linked to the *nonlinear* analysis of the classical, local kind. This is, perhaps, one of the most promising directions for future investigations.

The second one analyzes the nonlocal fields (i.e. ***p*-forms**), and as such is completely nonlocal: nonlocal fields, nonlocal analysis. Here, the concepts, and the machinery, are much different than those of local analysis. For one thing, the algebra of ***p*-forms** is a graded *noncommutative* ring. (This fact is fruitfully exploited in the third part of [NM], described in the next section.) Among other things, we study linear operators, and a detailed theory of **elliptic** operators is obtained.

3. GEOMETRO-ANALYSIS

The theories and concepts mentioned in the preceding sections are put together to build nonlocal-noncommutative analogs of several classical theories such as homology, cohomology, Hodge-de Rham, Chern-Weil, ***K*-theory**, index theory and such ([NM], Part III). We re-emphasize that all these theories involve functors with values in the category of *noncommutative* rings. Naturally, these theories encode much more information than do their classical, commutative, counterparts. For each one of these theories, there is a natural transformation from the noncommutative to the commutative case. Furthermore, these natural transformations form commutative diagrams along with other internal natural transformations of each theory. For example, the Chern characters of nonlocal and local types form a commutative square of natural transformations along with those relating the respective ***K*-functors** and the ***H*-functors**, i.e. the cohomology theories. This network of natural transformations allow lifting problems—and often entire solutions—from the commutative to the noncommutative functors on one hand while allowing translation of noncommutative problems into their commutative analogs on the other. Of course, this latter is more convenient in answering questions which have negative answers.

4. ALGEBRAIC GEOMETRY

The nonlocal methods can be easily extended to algebraic geometry and analytic geometry: we have nonlocal sections of schemes giving noncommutative rings. Thus, it becomes possible to analyze the geometry of noncommutative rings/schemes using commutative schemes and sheaves. On the other hand, nonlocal methods can shed more light on commutative schemes. Thus, we have noncommutative Chow ring of a scheme, noncommutative Grothendieck ring, nonlocal

Grothendieck-Riemann-Roch, etc. ([NM], Part IV). Here, too, we have a set of natural transformations from the noncommutative functors to their classical commutative namesakes.

5. SEVERAL COMPLEX VARIABLES

The remarks of last section also apply to the category of analytic spaces and the associated sheaves of modules. Nonlocal cohomology of sheaves provide a much wider context in which to explore classical theories, such as Oka-Cartan theory and their extensions/applications. Again, we have a natural transformation from the noncommutative to the commutative cohomology ([NM], Part V).

6. NUMBER THEORY

This is, perhaps, the most difficult part of mathematics, and the nonlocal viewpoint promises to be quite an addition to its already spectacular arsenal. A lot of the arithmetic geometry has nonlocal extensions, not unlike the cases of algebraic and analytic geometries discussed above. This is the field the present author can not even pretend to be any proficient in; this is good news, because the hope is that other truly qualified practitioners of this field will find many more uses of the nonlocal viewpoint. Nevertheless, we mention nonlocal Arakelov theory, and nonlocal arithmetic Riemann-Roch.

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